

# Design Of Inspection And Maintenance Models Based On The CCC-chart

Ling-Yau Chan · Associate Professor · Hong Kong

Key Words: Inspection, maintenance, statistical process control, economic design.

## SUMMARY & CONCLUSION

In this research, six maintenance models are constructed based on whether minor inspection, major inspection, minor maintenance and major maintenance are performed on a system. The system to study is a production process in which items produced can be classified as either conforming or nonconforming, and a statistical process control chart called CCC-chart (cumulative count control chart) can be applied to monitor the process. The maintenance models are analyzed quantitatively, and selection of models can be based on an economic consideration. The total cost can be broken down into inspection cost, maintenance cost, and the cost due to deterioration of the process. From the analytic results obtained, the choice of maintenance plan can be optimized from an economic point of view.

## 2. INTRODUCTION

In the industry, decisions always have to be made on when to perform maintenance for a system, what type of maintenance work to be performed, or whether to perform maintenance at all. If inspection of the system and minor maintenance are performed at suitable times, further deterioration of the system can be prevented, and sometimes expensive major maintenance can be avoided. However, if inspection and minor maintenance work are performed more frequently than needed, not only will the system be disturbed from time to time and unnecessary costs will be incurred, but additional causes of failure may also be introduced. If maintenance cost is high compared to the failure cost of the system, the system administrator may decide not to perform maintenance at all, but wait until the system fails, and then perform major maintenance on the whole system. Such a practice is not uncommon in the industry.

Statistical process control charts can be used to assist decision making in maintenance ([1]). In this research, ge-

ometric random variable is used in the modeling, and the CCC-chart is used to monitor the system. In Section 3, signals that indicate deterioration of the system, forms of inspection for the system, and types of maintenance will be defined. In Section 4, six maintenance models will be established. In Section 5, the six maintenance models will be analyzed quantitatively. In section 6, different costs including penalty cost due to production of nonconforming items, costs for inspection and maintenance of the system are taken into account, and maintenance plans are optimized in terms of the total expected cost. Some examples will be given. Section 7 contains some discussion.

## 3. STATES OF THE SYSTEM

Suppose that the system under consideration is a production process, in which items produced can be classified as conforming or nonconforming. Assume that the process has three states, in which  $S_0$  is its normal state. The process will deteriorate to state  $S_1$  as time passes. When the process is in state  $S_1$ , it will eventually deteriorates to state  $S_2$  if maintenance is not performed. We shall use the abbreviations  $m_1$  and  $m_2$  to denote "minor maintenance" and "major maintenance", respectively. It is assumed that  $m_1$  is less costly to carry out, but is not as thorough as  $m_2$ . The following will be assumed.

1. When the process is in state  $S_0$ , no maintenance is required, and neither  $m_1$  nor  $m_2$  will change its state.
2. When the process is in  $S_0$ , at any instant it will deteriorate to state  $S_1$  with a positive probability  $\pi_{01}$ .
3. When the process is in state  $S_1$ , either  $m_1$  or  $m_2$  will restore the process back to state  $S_0$ .
4. When the process is in state  $S_1$ , at any instant it will deteriorate to state  $S_2$  with a positive probability  $\pi_{12}$ .
5. When the process is in state  $S_2$ ,  $m_1$  has no effect on it, but  $m_2$  will restore the process back to state  $S_0$ .
6. The process will not deteriorate directly from state  $S_0$  to state  $S_2$  without going through state  $S_1$ .

Suppose that two types of signals that indicate deterioration of the process,  $s_1$  and  $s_2$ , will appear from time to time. When the process is in state  $S_0$ , the probability of occurrence of  $s_1$  or  $s_2$  will be small. When the process is in state  $S_1$ , the probability of occurrence of  $s_1$  is large, and when the process is in state  $S_2$ , the probability of occurrence of  $s_2$  is large.

In order that probabilities for occurrence of  $s_1$  and  $s_2$  can be calculated, a statistical model must be assumed. Here we apply a geometric random variable to describe the production process in which items are produced one after another, and the probability for an item produced to be nonconforming is  $p$ . The number of items inspected,  $n$ , until a nonconforming item is observed, is a geometric random variable ([2]) with the following probability function  $f(n)$  and cumulative distribution function  $F(n)$  ( $n = 1, 2, \dots$ ):

$$f(n) = p(1-p)^{n-1}, \quad F(n) = 1 - (1-p)^n. \quad (3.1)$$

A CCC-chart ([3], [4], [5], [6]) for detecting upward shift of  $p$  (deterioration of the process) has a lower control limit  $n_L$  for  $n$ , which corresponds to a certain probability of false alarm  $\alpha$ . It follows from (3.1) that

$$F(n_L) = 1 - (1-p)^{n_L} = \alpha, \quad (3.2)$$

$$n_L = \log(1-\alpha)/\log(1-p). \quad (3.3)$$

Divide the set of all positive integers  $Z^+$  into the following three sets:

$$Z_0 = \{n \in Z^+ : n_1 + 1 \leq n < \infty\},$$

$$Z_1 = \{n \in Z^+ : n_2 + 1 \leq n \leq n_1\},$$

$$Z_2 = \{n \in Z^+ : 1 \leq n \leq n_2\},$$

where  $1 \leq n_2 < n_1$  are given integers. Occurrence of a nonconforming item at  $n \in Z_i$  ( $i = 0, 1, 2$ ) is defined as a type  $i$  signal.

A signal  $s_0$  indicate that the process is in state  $S_0$ , and such a signal does not call for any action. When a signal  $s_1$  or  $s_2$  appears, either minor inspection,  $i_1$ , or major inspection,  $i_2$  may be carried out. Assume that  $i_2$  is a thorough inspection, while  $i_1$  is a superficial one which is less costly to carry out than  $i_2$ . The following three arrangements,  $I_{1+2}$ ,  $I_2$  and  $I_0$ , are alternative inspection strategies:

(1)  $I_{1+2}$ : Both  $i_1$  and  $i_2$  are employed.

In this arrangement, an  $s_1$  will trigger  $i_1$ , and an  $s_2$  will trigger  $i_2$  for the process. When the process is in state  $S_0$  or  $S_1$ , both  $i_1$  and  $i_2$  will correctly reveal the true state

of the process. When the process is in state  $S_2$ , however,  $i_1$  will incorrectly indicate that the process is in state  $S_1$ , and only  $i_2$  will correctly reveal that the process is in state  $S_2$ .

(2)  $I_2$ : Only  $i_2$  is employed.

In this arrangement, either  $s_1$  or  $s_2$  will trigger  $i_2$  which will always correctly reveal the true state of the process, no matter whether the process is in state  $S_0$ ,  $S_1$  or  $S_2$ .

(3)  $I_0$ : No inspection.

In this arrangement, no inspection will be carried out, even when  $s_1$  or  $s_2$  appears.

Whether to adopt strategies  $I_{1+2}$  or  $I_2$  depends on how much more difficult or costly it is to carry out  $i_2$  than  $i_1$ , and the consequences of wrong indication of the state of the process. The arrangement  $I_0$  of no inspection will be suitable when inspecting the process is either too costly or not feasible.

In the next section, these three arrangements for inspection will be combined with three arrangements for maintenance to form six maintenance models.

#### 4. MAINTENANCE MODELS

A cycle of the process is said to be completed when the process starts at state  $S_0$ , deteriorates to state  $S_1$  or state  $S_2$ , and finally restored back to state  $S_0$  by  $m_1$  or  $m_2$ . In what follows, three maintenance arrangements,  $M_{1+2}$ ,  $M_2$  and  $M_0$ , will be considered. In  $M_{1+2}$ , both  $m_1$  and  $m_2$  will be carried out; in  $M_2$ , only  $m_2$  will be carried out, but not  $m_1$ ; and in  $M_0$ , no maintenance work will be carried out. Under  $M_{1+2}$  or  $M_2$ , the process will eventually complete a cycle. Under  $M_0$ , no maintenance work will be carried out, and after the process has started it will eventually change to state  $S_2$  and remain in state  $S_2$ . Combining  $I_{1+2}$ ,  $I_2$ ,  $I_0$  with  $M_{1+2}$ ,  $M_2$ ,  $M_0$  in various ways produces the following six maintenance models.

(1) Model ( $I_{1+2}$ ,  $M_{1+2}$ ).

In this model, all of  $i_1$ ,  $i_2$ ,  $m_1$  and  $m_2$  will be carried out. Three possible scenarios under events  $A_I$ ,  $B_I$ ,  $C_I$  are depicted in Figure 1. In Figure 1, the events enclosed in each pair of square brackets occur in sequence,  $(p_i)$  means that the process is in state  $S_i$  ( $i = 0, 1, 2$ ), and three mutually exclusive events  $A_I$ ,  $B_I$ ,  $C_I$  are defined by:

$A_I$ : An  $s_1$  appears when the process is in state  $S_1$ .

$B_I$ : An  $s_2$  appears when the process is in state  $S_1$ .

$C_I$ : The process changes from state  $S_1$  to state  $S_2$  before any  $s_1$  or  $s_2$  appears.

The signals  $s_0$ 's are not indicated in Figure 1 (nor in

all other figures below), since such signals do not call for any action. After event  $A_I$  or  $B_I$  has occurred,  $m_1$  or  $m_2$  will bring the process back to state  $S_0$ . After event  $C_I$  has occurred and the process is in state  $S_2$ ,  $m_1$  will not change the state of the process, but  $m_2$  will bring the process back to state  $S_0$ .

$$\begin{aligned}
(p_0) &\rightarrow (p_1) \rightarrow \\
A_I: [s_1, i_1, m_1] &\rightarrow (p_0) \\
B_I: [s_2, i_2, m_1] &\rightarrow (p_0) \\
C_I: &\rightarrow (p_2) \rightarrow [s_1, i_1, m_1] \rightarrow \dots \rightarrow \\
&[s_1, i_1, m_1] \rightarrow [s_2, i_2, m_2] \rightarrow (p_0)
\end{aligned}$$

Figure 1. Scenarios for the process to complete a cycle under model  $(I_{1+2}, M_{1+2})$ .

#### (2) Model $(I_2, M_{1+2})$ .

In this model,  $i_2$ ,  $m_1$  and  $m_2$  will be carried out, but not  $i_1$ . This model will be suitable when the effort required to carry out  $i_1$  is about the same as that for  $i_2$ . Here either an  $s_1$  or  $s_2$  will be followed by  $i_2$ . Three possible scenarios under events  $A_I$ ,  $B_I$ ,  $C_I$  defined above are depicted in Figure 2.

$$\begin{aligned}
(p_0) &\rightarrow (p_1) \rightarrow \\
A_I: [s_1, i_2, m_1] &\rightarrow (p_0) \\
B_I: [s_2, i_2, m_1] &\rightarrow (p_0) \\
C_I: &\rightarrow (p_2) \rightarrow [s_1 \text{ or } s_2, i_2, m_2] \rightarrow (p_0)
\end{aligned}$$

Figure 2. Scenarios for the process to complete a cycle under model  $(I_2, M_{1+2})$ .

#### (3) Model $(I_0, M_{1+2})$ .

In this model, no inspection will be carried out. This model may be applied when inspection is either too costly or not feasible. When an  $s_1$  occurs,  $m_1$  will be carried out immediately, and when an  $s_2$  occurs,  $m_2$  will be carried out immediately. In this model, the three possible scenarios under events  $A_I$ ,  $B_I$ ,  $C_I$  defined above are depicted in Figure 3.

$$\begin{aligned}
(p_0) &\rightarrow (p_1) \rightarrow \\
A_I: [s_1, m_1] &\rightarrow (p_0) \\
B_I: [s_2, m_2] &\rightarrow (p_0) \\
C_I: &\rightarrow (p_2) \rightarrow [s_1, m_1] \rightarrow \dots \rightarrow [s_1, m_1] \rightarrow \\
&[s_2, m_2] \rightarrow (p_0)
\end{aligned}$$

Figure 3. Scenarios for the process to complete a cycle under model  $(I_0, M_{1+2})$ .

#### (4) Model $(I_2, M_2)$ .

In this model, when an  $s_2$  occurs,  $i_2$  will be carried out, and if  $i_2$  reveals that the process is in state  $S_2$ ,  $m_2$  will

be carried out. Here  $m_1$  will not be carried out, all  $s_1$ 's will be ignored. This model will be suitable when the effort required to carry out  $i_1$  and  $m_1$  is nearly as much as those required for  $i_2$  and  $m_2$ . Figure 4 depicts two possible scenarios of this model. Two mutually exclusive events  $B_{II}$  and  $C_{II}$  are defined by:

$B_{II}$ : An  $s_2$  occurs, or some  $s_1$ 's occur and then follows by an  $s_2$ .

$C_{II}$ : The process changes from state  $S_1$  to state  $S_2$  before any  $s_2$  occurs.

$$\begin{aligned}
(p_0) &\rightarrow (p_1) \rightarrow \\
B_{II}: [s_1, \dots, s_1, s_2, i_2, m_2] &\rightarrow (p_0) \\
C_{II}: &\rightarrow (p_2) \rightarrow [s_1, \dots, s_1, s_2, i_2, m_2] \rightarrow (p_0)
\end{aligned}$$

Figure 4. Scenarios for the process to complete a cycle under model  $(I_2, M_2)$ .

#### (5) Model $(I_0, M_2)$ .

In this model, no inspection and no  $m_1$  will be carried out. All  $s_1$ 's will be ignored. When an  $s_2$  occurs,  $m_2$  will be carried out immediately without any inspection. This model can be represented by Figure 4 with the  $i_2$ 's removed.

#### (6) Model $(I_0, M_0)$ .

In this model, neither inspection nor maintenance will be carried out. This model suits the case when the effort required to carry out inspection and maintenance are costly compared with the penalty cost due to production of nonconforming items. Figure 5 shows the model.

$$(p_0) \rightarrow (p_1) \rightarrow (p_2)$$

Figure 5. Scenarios for the process to complete a cycle under model  $(I_0, M_0)$ .

In the above six models,  $(I_{1+2}, M_{1+2})$  and  $(I_0, M_{1+2})$  depend on both  $n_1$  and  $n_2$ ,  $(I_2, M_{1+2})$  depends on  $n_1$  but not  $n_2$ , models  $(I_2, M_2)$  and  $(I_0, M_2)$  depend on  $n_2$  but not  $n_1$ , and  $(I_0, M_0)$  is independent of both  $n_1$  and  $n_2$ . Table 1 shows these models; the combinations that are not applicable are indicated by "N.A.". For example, if minor maintenance is not to be carried out, there will no point to carry out minor inspection, and therefore the model  $(I_{1+2}, M_2)$  is not applicable, and so is the model  $(I_{1+2}, M_0)$ , and so on.

In the next section, analytic expressions for the probabilities of events and average number of items inspected will be established.

Table 1. Maintenance models

		$M_{1+2}$	$M_2$	$M_0$
$I_{1+2}$	<i>Inspection:</i>	1 and 2	N.A.	N.A.
	<i>Maintenance:</i>	1 and 2	N.A.	N.A.
$I_2$	<i>Inspection:</i>	2 only	2 only	N.A.
	<i>Maintenance:</i>	1 and 2	2 only	N.A.
$I_0$	<i>Inspection:</i>	nil	nil	nil
	<i>Maintenance:</i>	1 and 2	2 only	nil

## 5. QUANTITATIVE ANALYSIS

Let  $\xi_0$  denote the instant of time immediately after the process has started, or immediately after the occurrence of a nonconforming item when the process is in state  $S_0$ . Let  $Q_{0(i)}$  ( $i = 0, 1, 2$ ) be the probability for a type  $i$  signal to appear when the process is in state  $S_0$ . Since the number of items inspected to observe a nonconforming item is a geometric distribution, we have

$$Q_{0(0)} = \sum_{j \in Z_0} p_0(1-p_0)^{j-1} = (1-p_0)^{n_1}, \quad (5.1)$$

$$\begin{aligned} Q_{0(1)} &= \sum_{j \in Z_1} p_0(1-p_0)^{j-1} \\ &= (1-p_0)^{n_2} - (1-p_0)^{n_1}, \end{aligned} \quad (5.2)$$

$$Q_{0(2)} = \sum_{j \in Z_2} p_0(1-p_0)^{j-1} = 1 - (1-p_0)^{n_2}. \quad (5.3)$$

Let  $\xi_1$  denote the instant of time immediately after the state of the process has shifted from  $S_0$  to  $S_1$ , or immediately after the occurrence of a nonconforming item when the process is in state  $S_1$ . Let  $Q_{1(i)}$  ( $i = 0, 1, 2$ ) be the probability of observing the first nonconforming item since  $\xi_1$ , and this nonconforming item gives a type  $i$  signal ( $i = 0, 1, 2$ ), while the process remains in state  $S_1$ . It can be proved that

$$\begin{aligned} Q_{1(0)} &= \sum_{j \in Z_0} p_1(1-p_1)^{j-1}(1-\pi_{12})^j \\ &= \frac{(1-\pi_{12})p_1((1-p_1)(1-\pi_{12}))^{n_1}}{1 - (1-p_1)(1-\pi_{12})}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} Q_{1(1)} &= \sum_{j \in Z_1} p_1(1-p_1)^{j-1}(1-\pi_{12})^j \\ &= \frac{(1-\pi_{12})p_1[(1-p_1)(1-\pi_{12})]^{n_2}}{1 - (1-p_1)(1-\pi_{12})} \\ &\quad - \frac{(1-\pi_{12})p_1[(1-p_1)(1-\pi_{12})]^{n_1}}{1 - (1-p_1)(1-\pi_{12})}, \end{aligned} \quad (5.5)$$

$$Q_{1(2)} = \sum_{j \in Z_2} p_1(1-p_1)^{j-1}(1-\pi_{12})^j$$

$$= \frac{(1-\pi_{12})p_1[1 - ((1-p_1)(1-\pi_{12}))^{n_2}]}{1 - (1-p_1)(1-\pi_{12})}. \quad (5.6)$$

The probability of observing a shift of the process from state  $S_1$  to state  $S_2$  since  $\xi_1$  is

$$\begin{aligned} Q_{1,2} &= \sum_{j=1}^{\infty} \pi_{12}(1-p_1)^{j-1}(1-\pi_{12})^{j-1} \\ &= \frac{\pi_{12}}{1 - (1-p_1)(1-\pi_{12})}. \end{aligned} \quad (5.7)$$

Let  $\xi_2$  denote the instant of time immediately after the state of the process has shifted from  $S_1$  to  $S_2$ , or immediately after the occurrence of a nonconforming item when the process is in state  $S_2$ . Let  $Q_{2(i)}$  ( $i = 0, 1, 2$ ) be the probability of observing the first nonconforming item since  $\xi_2$ , and this nonconforming item gives a type  $i$  signal ( $i = 0, 1, 2$ ), while the process remains in state  $S_2$ . We have

$$Q_{2(0)} = \sum_{j \in Z_0} p_2(1-p_2)^{j-1} = (1-p_2)^{n_1}, \quad (5.8)$$

$$Q_{2(1)} = \sum_{j \in Z_1} p_2(1-p_2)^{j-1} = (1-p_2)^{n_2} - (1-p_2)^{n_1}, \quad (5.9)$$

$$Q_{2(2)} = \sum_{j \in Z_2} p_2(1-p_2)^{j-1} = 1 - (1-p_2)^{n_2}. \quad (5.10)$$

The probability of occurrence of events  $A_I, B_I, C_I$  defined in models  $(I_{1+2}, M_{1+2})$ ,  $(I_2, M_{1+2})$  and  $(I_0, M_{1+2})$  are

$$P(A_I) = \sum_{i=1}^{\infty} Q_{1(0)}^{i-1} Q_{1(1)} = \frac{Q_{1(1)}}{1 - Q_{1(0)}}, \quad (5.11)$$

$$P(B_I) = \sum_{i=1}^{\infty} Q_{1(0)}^{i-1} Q_{1(2)} = \frac{Q_{1(2)}}{1 - Q_{1(0)}}, \quad (5.12)$$

$$P(C_I) = \sum_{i=1}^{\infty} Q_{1(0)}^{i-1} Q_{1,2} = \frac{Q_{1,2}}{1 - Q_{1(0)}}, \quad (5.13)$$

respectively.

The probability of occurrence of events  $B_{II}, C_{II}$  defined in models  $(I_2, M_2)$  and  $(I_0, M_{1+2})$  are

$$\begin{aligned} P(B_{II}) &= \sum_{i=1}^{\infty} (Q_{1(0)} + Q_{1(1)})^{i-1} Q_{1(2)} \\ &= \frac{Q_{1(2)}}{1 - Q_{1(0)} - Q_{1(1)}}, \end{aligned} \quad (5.14)$$

$$P(C_{II}) = \sum_{i=1}^{\infty} (Q_{1(0)} + Q_{1(1)})^{i-1} Q_{1,2}$$

$$= \frac{Q_{1,2}}{1 - Q_{1(0)} - Q_{1(1)}}. \quad (5.15)$$

We shall find the expected items inspected for the process to complete a cycle. Let  $\delta = (1 - p_1)(1 - \pi_{12})$ , and define

$$\begin{aligned} L_0 &= \sum_{j \in Z_0} jp_1(1 - p_1)^{j-1}(1 - \pi_{12})^j \\ &= \frac{p_1(1 - \pi_{12})[\delta^{n_1}(n_1 + 1 - n_1\delta)]}{(1 - \delta)^2}, \end{aligned} \quad (5.16)$$

$$\begin{aligned} L_1 &= \sum_{j \in Z_1} jp_1(1 - p_1)^{j-1}(1 - \pi_{12})^j \\ &= \frac{p_1(1 - \pi_{12})[\delta^{n_2}(n_2 + 1 - n_2\delta)]}{(1 - \delta)^2} \\ &\quad - \frac{p_1(1 - \pi_{12})[\delta^{n_1}(n_1 + 1 - n_1\delta)]}{(1 - \delta)^2}, \end{aligned} \quad (5.17)$$

$$\begin{aligned} L_2 &= \sum_{j \in Z_2} jp_1(1 - p_1)^{j-1}(1 - \pi_{12})^j \\ &= p_1(1 - \pi_{12}) \frac{1 - \delta^{n_2}(n_2 + 1 - n_2\delta)}{(1 - \delta)^2}. \end{aligned} \quad (5.18)$$

$$\begin{aligned} L_3 &= \sum_{j=1}^{\infty} (j-1)\pi_{12}(1 - p_1)^{j-1}(1 - \pi_{12})^{j-1} \\ &= \pi_{12}\delta/(1 - \delta)^2, \end{aligned} \quad (5.19)$$

$$\begin{aligned} L_4 &= \sum_{j \in Z_0 \cup Z_1} jp_2(1 - p_2)^{j-1} \\ &= [(1 - p_2)^{n_2}(n_2 + 1 - n_2(1 - p_2))]/p_2, \end{aligned} \quad (5.20)$$

$$\begin{aligned} L_5 &= \sum_{j \in Z_2} jp_2(1 - p_2)^{j-1} \\ &= \frac{1 - (1 - p_2)^{n_2}(n_2 + 1 - n_2(1 - p_2))}{p_2}. \end{aligned} \quad (5.21)$$

The following Propositions and Lemmas can be proved. However, their proofs will be omitted here.

**Proposition I.** For models  $(I_{1+2}, M_{1+2})$  or  $(I_0, M_{1+2})$ , the average number of items inspected,  $ANI_I$ , for the process to complete a cycle which starts from state  $S_0$ , deteriorates to state  $S_1$  or state  $S_2$ , and finally be restored back to state  $S_0$  is

$$\begin{aligned} ANI_I &= \frac{1}{\pi_{01}} + \frac{1 - \pi_{12}}{\pi_{12} + p_1(1 - \pi_{12})} \times \frac{1}{1 - Q_{1(0)}} \\ &\quad + \frac{Q_{1,2}}{1 - Q_{1(0)}} \times \frac{1}{p_2 Q_{2(2)}}. \end{aligned}$$

**Proposition II.** For model  $(I_2, M_{1+2})$ , the average number of items inspected,  $ANI_{II}$ , for the process to complete a cycle which starts from state  $S_0$ , deteriorates to

state  $S_1$  or state  $S_2$ , and finally be restored back to state  $S_0$  is

$$\begin{aligned} ANI_{II} &= \frac{1}{\pi_{01}} + \frac{1 - \pi_{12}}{\pi_{12} + p_1(1 - \pi_{12})} \times \frac{1}{1 - Q_{1(0)}} \\ &\quad + \frac{Q_{1,2}}{1 - Q_{1(0)}} \times \frac{1}{p_2(1 - Q_{2(0)})}. \end{aligned}$$

**Proposition III.** For models  $(I_2, M_2)$  and  $(I_0, M_2)$ , the average number of items inspected,  $ANI_{III}$ , for the process to complete a cycle which starts from state  $S_0$ , deteriorates to state  $S_1$  or state  $S_2$ , and finally be restored back to state  $S_0$  is

$$\begin{aligned} ANI_{III} &= \frac{1}{\pi_{01}} + \frac{1 - \pi_{12}}{\pi_{12} + p_1(1 - \pi_{12})} \times \frac{1}{1 - Q_{1(0)} - Q_{1(1)}} \\ &\quad + \frac{Q_{1,2}}{1 - Q_{1(0)} - Q_{1(1)}} \times \frac{1}{p_2 Q_{2(2)}}. \end{aligned}$$

Note that (5.4), (5.5), (5.7), (5.10) show that  $ANI_I$  depends on both  $n_1$  and  $n_2$ ,  $ANI_{II}$  depends on  $n_1$  but not  $n_2$ , and  $ANI_{III}$  depends on  $n_2$  but not  $n_1$ . The following Lemmas 1 – 11 are required in the proof of Propositions I – III. All the detailed proofs, however, will be omitted here.

**Lemma 1.** The expected number of items inspected since the process starts at state  $S_0$  until it shifts to state  $S_1$ , is

$$E_1 = \sum_{j=1}^{\infty} j\pi_{01}(1 - \pi_{01})^{j-1} = \frac{1}{\pi_{01}}.$$

**Lemma 2.** Under event  $A_I$ , the expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until a type 1 signal appears while the process still remains in state  $S_1$ , without any type 2 signal appearing before this type 1 signal, is

$$E_2 = \frac{Q_{1(1)}L_0}{(1 - Q_{1(0)})^2} + \frac{L_1}{1 - Q_{1(0)}}.$$

**Lemma 3.** Under event  $B_I$ , the expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until a type 2 signal appears while the process still remains in state  $S_1$ , without any type 1 signal appearing before this type 2 signal, is

$$E_3 = \frac{Q_{1(2)}L_0}{(1 - Q_{1(0)})^2} + \frac{L_2}{1 - Q_{1(0)}}.$$

**Lemma 4.** Under event  $C_I$ , the expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until it shifts to state  $S_2$ , without any type 1 or type 2 signal appearing during this period of time, is

$$E_4 = \frac{Q_{1,2}L_0}{(1 - Q_{1(0)})^2} + \frac{L_3}{1 - Q_{1(0)}}.$$

**Lemma 5.** Under event  $B_{II}$ , the expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until a type 2 signal appears, while the process still remains in state  $S_1$ , is

$$E_5 = \frac{Q_{1(2)}(L_0 + L_1)}{(1 - Q_{1(0)} - Q_{1(1)})^2} + \frac{L_2}{1 - Q_{1(0)} - Q_{1(1)}}.$$

Here some type 0 or type 1 signal may appear before this type 2 signal.

**Lemma 6.** Under event  $C_{II}$ , the expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until it shifts to state  $S_2$ , without any type 2 signal appearing during this period of time, is

$$E_6 = \frac{Q_{1,2}(L_0 + L_1)}{(1 - Q_{1(0)} - Q_{1(1)})^2} + \frac{L_3}{1 - Q_{1(0)} - Q_{1(1)}}.$$

**Lemma 7.** The expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_1$  to state  $S_2$ , until a type 2 signal occurs, is

$$E_7 = \frac{Q_{2(2)}L_4}{(1 - Q_{2(0)} + Q_{2(1)})^2} + \frac{L_5}{1 - Q_{2(0)} + Q_{2(1)}} \\ = \frac{1}{p_2 Q_{2(2)}}.$$

**Lemma 8.** The expected number of type  $i$  ( $i = 0, 1, 2$ ) signals that appear after the process has started, until just before the process shifts from state  $S_0$  state  $S_1$ , is

$$E_{8,i} = \frac{p_0 Q_{0(i)}}{\pi_{01}} \quad (i = 0, 1, 2).$$

**Lemma 9.** The expected number of type 1 signals that appear immediately after the process has shifted from state  $S_1$  to state  $S_2$ , until just before a type 2 signal occurs, is

$$E_9 = \frac{Q_{2(1)}}{Q_{2(2)}}.$$

**Lemma 10.** The expected number of items inspected immediately after the process has shifted from state  $S_1$  to state  $S_2$ , until either a type 1 or type 2 signal appears, is

$$E_{10} = \frac{(Q_{2(1)} + Q_{2(2)})L_4}{(1 - Q_{2(0)})^2} + \frac{L_5}{1 - Q_{2(0)}} \\ = \frac{1}{p_2(1 - Q_{2(0)})}.$$

**Lemma 11.** The expected number of items inspected from the instant of time immediately after the process has shifted from state  $S_0$  to state  $S_1$ , until it shifts to state  $S_2$ , is

$$E_{11} = \sum_{j=1}^{\infty} j \pi_{12} (1 - \pi_{12})^{j-1} = \frac{1}{\pi_{12}}.$$

## 6. ECONOMIC DESIGN

In reality, various factors such as availability of resources, operational convenience, loss due to process deterioration, and others, determine whether or not to perform maintenance on the process and how frequently should maintenance be performed. Generally speaking, inspection and maintenance keep the process in good shape and prevents unexpected increase of fraction of nonconforming items produced. Therefore, from a economic point of view, inspection and maintenance should be carried out if the benefit achieved is more than the loss due to the nonconforming items produced. In what follows, different costs incurred for the different maintenance models in Table 1 will be calculated.

The following three types of cost will be considered ([7], [8]): (1) the penalty cost due to the nonconforming items produced, (2) the cost spent in inspecting the process, and (3) the cost spent in carrying out maintenance work. Let

$c_{nc}$  = the penalty cost incurred when a nonconforming item is produced,

$c_{inv,1}$  = cost of carrying out  $i_1$  each time,

$c_{inv,2}$  = cost of carrying out  $i_2$  each time ( $c_{inv,1} \leq c_{inv,2}$ ),

$c_{m,1}$  = cost of carrying out  $m_1$  each time,

$c_{m,2}$  = cost of carrying out  $m_2$  each time ( $c_{m,1} \leq c_{m,2}$ ).

Let  $\bar{c}$  by the average total cost per item produced. For the maintenance models  $(I_{1+2}, M_{1+2})$  and  $(I_0, M_{1+2})$ ,  $\bar{c}$  will the sum of all the costs divided  $ANI_I$  in Proposition I. For model  $(I_2, M_{1+2})$ ,  $\bar{c}$  will the sum of all the expected costs divided  $ANI_{II}$  given in Proposition II. For models  $(I_2, M_2)$  and  $(I_0, M_2)$ ,  $\bar{c}$  will the sum of all the expected costs divided  $ANI_{III}$  in Proposition III. As for model  $(I_0, M_0)$ , since the process will change to state  $S_2$  and remain

there indefinitely, the cycle time is infinite and therefore

$$\bar{c} = \lim_{j \rightarrow \infty} \frac{p_0 E_1 c_{nc} + p_1 E_{11} c_{nc} + j p_2 c_{nc}}{E_1 + E_{11} + j} = p_2 c_{nc}.$$

For each maintenance models,  $\bar{c}$  can be minimized with respect to  $n_1$  and  $n_2$ . Then the minimum of  $\bar{c}$  for the six maintenance models can be compared, and the most cost effective model with the minimum  $\bar{c}$ , say  $\bar{c}_{min}$ , can be selected. The numerical results in Table 8(a)–(e) show that for different values of  $p_0, p_1, p_2, \pi_{01}, \pi_{12}, c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}$  and  $c_{m,2}$ , the minimum average cost  $\bar{c}_{min}$  can be attained under each of the five models  $(I_{1+2}, M_{1+2})$ ,  $(I_2, M_{1+2})$ ,  $(I_0, M_{1+2})$ ,  $(I_0, M_2)$  and  $(I_0, M_0)$ .

As for the model  $(I_2, M_2)$ , the fact is that given any set of values  $p_0, p_1, p_2, \pi_{01}, \pi_{12}, c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}$  and given any maintenance plan under this model, there always exists a less costly maintenance plan under model  $(I_2, M_{1+2})$ , provided that  $c_{m,1} < c_{m,2}$ . To see this, suppose that there is a maintenance plan under model  $(I_2, M_2)$ , called Plan 2, in which  $n_2 = n_0 > 1$ . Consider a maintenance plan under model  $(I_2, M_{1+2})$  with  $n_1 = n_0$  and  $n_2$  equal to any positive integer less than  $n_0$ , which will be called Plan 1. In Plan 2,  $m_2$  will be carried out even though only  $m_1$  is needed, but in Plan 1 both  $m_1$  and  $m_2$  are available and thus the unnecessary  $m_2$  can be avoided. Since  $c_{m,1} < c_{m,2}$ , Plan 2 is more costly than Plan 1. This can be proved quantitatively by noting that  $Z_1 \cup Z_2$  and  $Q_{\ell(1)} + Q_{\ell(2)}$  of Plan 1 are identical to  $Z_2$  and  $Q_{\ell(2)}$  of Plan 2 ( $\ell = 0, 1, 2$ ). Thus,  $P(A_I) + P(B_I)$  and  $P(C_I)$  for Plan 1 are identical to  $P(B_{II})$  and  $P(C_{II})$  for Plan 2, and so are the corresponding costs. Since  $c_{m,1} < c_{m,2}$ , maintenance cost for Plan 1 is less than that of Plan 2, and so is the total cost.

The argument in last paragraph is based on the assumption that  $c_{m,2}$  is the same for both Plan 1 and Plan 2. However, in some situations, an additional cost  $\Delta$  per maintenance task is required in order to maintain two maintenance procedures (minor and major), rather than just one (major). If  $\Delta$  is to be absorbed in the maintenance cost, then the cost of carrying out  $m_2$  each time under model  $(I_2, M_{1+2})$  may be  $c_{m,2} + \Delta$ , which is larger than the cost  $c_{m,2}$  required under model  $(I_2, M_2)$ . In this case, it is possible that  $\bar{c}_{min}$  under model  $(I_2, M_2)$  will be smaller than that under model  $(I_2, M_{1+2})$ , as illustrated in the numerical example in Table 2(f). This idea can be extended to the situation when additional cost is incurred in order to maintain two inspection procedures (minor and major), instead of just one (major), but the details will not be elaborated here.

It is therefore possible that  $\bar{c}_{min}$  can be achieved by any of the six maintenance models  $(I_{1+2}, M_{1+2})$ ,  $(I_2, M_{1+2})$ ,  $(I_0, M_{1+2})$ ,  $(I_2, M_2)$ ,  $(I_0, M_2)$  and  $(I_0, M_0)$ . This is illustrated by the numerical examples in Table 2, in which the  $\bar{c}_{min}$ 's are calculated based on  $(p_0, p_1, p_2, \pi_{01}, \pi_{12}) = (0.015, 0.019, 0.05, 0.0004, 0.0035)$ . For each data set, the minimum average cost  $\bar{c}_{min}$  among the six maintenance models is enclosed in brackets in Table 2.

Table 2. Numerical examples.

(a) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (120, 3, 18, 11, 22).						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	(1.891)	1.908	1.913	1.910	1.913	6
$n_2 =$	7	—	11	13	12	—
$n_1 =$	49	13	12	—	—	—
(b) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (120, 100, 108, 40, 220).						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	2.137	(2.121)	2.185	2.145	2.198	6
$n_2 =$	5	—	3	5	3	—
$n_1 =$	6	5	8	—	—	—
(c) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (120, 50, 108, 11, 22).						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	2.076	2.071	(1.913)	2.072	1.913	6
$n_2 =$	5	—	11	5	12	—
$n_1 =$	6	5	12	—	—	—
(d) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (120, 110, 120, 110, 120).						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	2.147	2.124	2.100	2.125	(2.081)	6
$n_2 =$	5	—	5	5	12	—
$n_1 =$	6	5	6	—	—	—
(e) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (1, 55, 65, 60, 70).						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	1.127	0.741	0.805	0.745	0.545	(0.5)
$n_2 =$	2	—	1	1	1	—
$n_1 =$	3	1	2	—	—	—
(f) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2})$ = (120, 5, 10, 11, 22). Here $\Delta = 350$ is introduced.						
	$I_{1+2}$ $M_{1+2}$	$I_2$ $M_{1+2}$	$I_0$ $M_{1+2}$	$I_2$ $M_2$	$I_0$ $M_2$	$I_0$ $M_0$
$\bar{c}_{min} =$	2.0149	2.0151	2.428	(1.888)	1.913	6
$n_2 =$	15	—	3	12	12	—
$n_1 =$	19	17	4	—	—	—

## 7. DISCUSSION

This research is concerned with design of maintenance strategy, for the case when minor and major signals of deterioration may appear from the system, and minor in-

spection, major inspection, minor maintenance and major maintenance may be performed on the system. It is assumed that major inspection can better reveal the state of the system but is more expensive than minor inspection, and major maintenance is more thorough but also more expensive than minor maintenance. Adequate inspection and maintenance is necessary, in order to maintain the reliability of the system. However, excess inspection and maintenance will not only disturb the process unnecessarily and but will also introduce additional chance of system failure. From an operational or economic point of view, there is an optimal maintenance strategy. In the optimal maintenance strategy, either both minor and major inspection and maintenance will be performed, only major inspection or major maintenance will be performed, or none of minor or major inspection, minor or major maintenance will be performed. In this research, this concept is applied on a production process in which the number of nonconforming items produced follows a geometric distribution, and a statistical control chart called CCC-chart is used to provide signals that indicate deterioration of the process. Optimal choice of maintenance strategy is based on an economic consideration.

In this research, it is assumed that the state of the process is discrete, and there are two types of inspection and two types of maintenance. Classifying a process into discrete states and assuming change of state of the process follows a memoryless distribution have been justified and commonly used in the literature (see for example, [9], [10], [11], [12], [13], [14]). However, the rationale in this research can be applied to a process whose states is measured on a continuous scale, and with more than two types of inspection and maintenance. If transition of state of the process is time dependent, more complicated statistical modeling is necessary, and distributions with changing failure rate (such as Weibull or nonhomogeneous Poisson distributions) may be appropriate tools.

## REFERENCES

1. A.Y. Cheng, R.Y. Liu, J.T. Luxhøj, "Thresholds for safety inspection measurements based on control charts", *International Journal of Reliability, Quality and Safety Engineering*, vol 42, 1997, pp 205–225.
2. W.H. Woodall, "Control charts based on attribute data: bibliography and review", *Journal of Quality Technology*, vol 29, 1997, pp 172–183.
3. T.W. Calvin, "Quality control techniques for 'zero-defects' ", *IEEE Transactions on Components, Hybrid and Manufacturing Technology*, vol CHMT-6, 1983, pp 323–328.
4. T.N. Goh, "A control chart for very high yield processes", *Quality Assurance* vol 13, 1987, pp 18–22.
5. L.Y. Chan, L.Y., M. Xie, T.N. Goh, "Two-stage control charts for high yield processes", *International Journal of Reliability, Quality and Safety Engineering*, vol 4, 1997,

pp 149–165.

6. L.Y. Chan, D.K.J. Lin, M. Xie, T.N. Goh, "Cumulative probability control charts for geometric and exponential process characteristics", *International Journal of Production Research*, vol 40, 2002, pp 133–150.
7. C. Ho, "Economic design of control charts: a literature review for 1981–1991", *Journal of Quality Technology*, vol 26, 1994, pp 39–53,
8. G. Tagaras, "A survey of recent developments in the design of adaptive control charts", *Journal of Quality Technology*, vol 30, 1998, pp 212–231.
9. W.K. Chiu, K.C. Cheung, "An economic design model for  $\bar{X}$  charts with warning limits", *Journal of Quality Technology*, vol 9, 1977, 166–171.
10. T.K. Das, V. Jain, A. Gosavi, "Economic design of dual-sampling-interval policies for  $\bar{X}$  charts with and without run rules", *IIE Transactions*, vol 29, 1977, 497–506.
11. T.K. Das, V. Jain, "An economic design model for  $\bar{X}$  charts with random sampling policies", *IIE Transactions*, vol 29, 1977, 507–518.
12. G.R. Gordon, J.I. Weindling, "A cost model for economic design of warning limit control chart schemes", *American Institute of Industrial Engineers Transactions*, vol 7, 1975, 319–329.
13. T.J. Lorenzen, L.C. Vance, "The economic design of control charts: a unified approach", *Technometrics*, vol 28, 1986, 3–10.
14. G. Tagaras and H.L. Lee, "Economic design of control charts with different control limits for different assignable causes", *Management Science*, vol 34, 1988, 1347–1366.

## BIOGRAPHY

Ling-Yau Chan, *PhD*

Department of Industrial and Manufacturing Systems Engineering  
The University of Hong Kong  
Pokfulam Road  
Hong Kong

e-mail: plychanhku.hk

Ling-Yau Chan has a BSc and an MPhil in mathematics, and a PhD in statistics. He is responsible for teaching of courses on mathematics, statistics, reliability and quality management in Engineering Faculty in The University of Hong Kong. He is a Fellow of The Royal Statistical Society, and a reviewer for *Mathematical Reviews*. His research areas include optimal experimental design, experiments with mixtures, reliability, statistical process control, optimization and quality management. He collaborates with scholars from different countries, and has published more than 60 papers in these research areas.